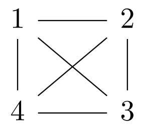
GROUP THEORY 2024 - 25, EXERCISE SHEET 2

Exercise 1. *Warm up exercise*
Consider the natural action of S_4 on the set X of all 2-elements subsets of $\{1,2,3,4\}$. This action can be seen as permuting the edges of the following diagram



(1) Determine the orbit of $\{1,2\} \in X$;
(2) Find the stabilizer of $\{1,2\} \in X$;
(3) Verify that the Orbit-Stabilizer theorem holds by recovering $|S_4|$ from the previous points.

Exercise 2. *Action on cosets*
Let G be a group and $H \leq G$ a subgroup. We define an action of G on G/H by

$$G \times G/H \to G/H$$
$$(g, g'H) \mapsto gg'H$$

(1) Show that this indeed defines a G-action;
(2) For $gH \in G/H$, find the stabilizer $\mathrm{Stab}_G(gH)$;

If X, Y are sets endowed with G-actions, we say that X and Y are isomorphic as G-sets if there exists a bijective function $f : X \to Y$ such that $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in X$.

Exercise 3. Let H and K be subgroups of a group G. Define the action of G on the sets G/H and G/K by left multiplication (see Exercise 2). Show that G/H and G/K are isomorphic as G-sets if and only if H and K are conjugate subgroups of G, i.e. there exists $g \in G$ such that $gHg^{-1} = K$.

Exercise 4. Let $V = \mathbb{F}_2^3$ and $G = GL(V)$ the group of linear automorphisms of V (its elements are bijective linear maps $V \to V$).

(1) Define a natural action of G on
$$X = \{W \subset V | \dim(W) = 2\}.$$
(2) Show the action defined above is transitive;
(3) Determine the cardinality $|X|$.

Exercise 5. (1) Let $G \leq S_n$. Consider its natural action on $\Omega = \{1, 2, \ldots, n\}$. Show that if G acts transitively on Ω, then n divides $|G|$.

(2) Let G be a group and $X = \{H \leq G\}$ the set of subgroups of G. Show that G acts on X by

$$G \times X \to X$$
$$(g, H) \mapsto gHg^{-1}.$$

For $H \in X$, what is the stabilizer $\mathrm{Stab}_G(H)$?

Exercise 6. *p-group actions and fixed points*
Use the orbit-stabiliser theorem to prove the following result. Let $p > 0$ be a prime number and let G be a group of order p^n for some $n \geq 1$ which acts on a set X with $p \nmid |X|$. Show that there exists $x \in X$ such that $g \cdot x = x$ for all $g \in G$.